

An explicit and positivity preserving numerical scheme for the mean reverting CEV model

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Abstract

In this paper we propose an explicit and positivity preserving scheme for the mean reverting CEV model which converges in the mean square sense with convergence order $a(a - 1/2)$.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a complete probability space with a filtration and let a Wiener process $(W_t)_{t \geq 0}$ defined on this space. We consider here the mean reverting CEV process,

$$x_t = x_0 + \int_0^t (kl - kx_s)ds + \sigma \int_0^t x_s^a dW_s, \quad (1)$$

where $k, l, \sigma \geq 0$ and $a \in (1/2, 1)$. It is well known that this sde has a unique strong solution which is strictly positive. Our starting point was the paper of [1] in which the author proposes an implicit and positivity preserving numerical scheme to approximate the above process. This stochastic process plays important role in financial mathematics. If one wants to use the above model to price complicate path-dependent options maybe it is useful to approximate it numerically. The usual Euler scheme (see [4]) does not preserve positivity and therefore numerical schemes with this feature are needed. In this direction one can see [1] and [3]. Our goal here is to present an explicit, positivity preserving numerical scheme that converges in the mean

square sense to the true solution with, at least, $a(a-1/2)$ order of convergence. There must be extensive numerical experiments to compare all these methods and decide which of them is the best in each set of parameters.

We will construct our scheme using the semi discrete method that we have proposed in [5] (and further extended in [6], [7]). Let $0 = t_0 < t_1 < \dots < t_n = T$ and set $\Delta = \frac{T}{n}$. Consider the following stochastic process

$$y_t = \left| \sigma(1-a)(W_t - W_{t_k}) + \left(y_{t_k}(1-k\Delta) + \Delta(kl - \frac{a\sigma^2 y_{t_k}^{2a-1}}{2}) \right)^{1-a} \right|^{\frac{1}{1-a}} = |z_t|^{\frac{1}{1-a}}, \quad (2)$$

for $t \in (t_k, t_{k+1}]$ where

$$z_t = \sigma(1-a)(W_t - W_{t_k}) + \left(y_{t_k}(1-k\Delta) + \Delta(kl - \frac{a\sigma^2 y_{t_k}^{2a-1}}{2}) \right)^{1-a},$$

for $t \in (t_k, t_{k+1}]$. To show that this stochastic process is well defined we will check whether

$$y_{t_k}(1-k\Delta) + \Delta(kl - \frac{a\sigma^2 y_{t_k}^{2a-1}}{2}) \geq 0.$$

For $kl \geq \frac{a\sigma^2}{2}$, $\Delta \leq \frac{1}{k + \frac{a\sigma^2}{2}}$ and noting that $0 < 2a-1 < 1$ we have

$$\begin{aligned} \left(y_{t_k}(1-k\Delta) + \Delta(kl - \frac{a\sigma^2 y_{t_k}^{2a-1}}{2}) \right) \mathbb{I}_{\{y_{t_k} > 1\}} + \left(y_{t_k}(1-k\Delta) + \Delta(kl - \frac{a\sigma^2 y_{t_k}^{2a-1}}{2}) \right) \mathbb{I}_{\{y_{t_k} \leq 1\}} &\geq \\ \mathbb{I}_{\{y_{t_k} > 1\}} y_{t_k}^{2a-1} \left(1 - k\Delta - \frac{\Delta a\sigma^2}{2} \right) + \Delta \left(kl - \frac{a\sigma^2}{2} \right) &\geq 0 \end{aligned}$$

Therefore we impose the following assumption.

Assumption A We assume that $x_0 \in \mathbb{R}_+$. Moreover, we suppose that

$$kl \geq \frac{a\sigma^2}{2}, \quad \Delta \leq \frac{2}{2k + a\sigma^2}.$$

This stochastic process, using Ito's formula, has the differential form,

$$y_t = y_{t_k}(1-k\Delta) + \Delta(kl - \frac{a\sigma^2 y_{t_k}^{2a-1}}{2}) + \int_{t_k}^t \frac{a\sigma^2}{2} y_s^{2a-1} ds + \sigma \int_{t_k}^t y_s^a \operatorname{sgn}(z_s) dW_s, \quad t \in (t_k, t_{k+1}],$$

Concluding, the numerical scheme that we propose for the mean reverting CEV model is the following,

$$y_{t_{k+1}} = \left| \sigma(1-a)(W_{t_{k+1}} - W_{t_k}) + \left(y_{t_k}(1-k\Delta) + \Delta(kl - \frac{a\sigma^2 y_{t_k}^{2a-1}}{2}) \right)^{1-a} \right|^{\frac{1}{1-a}}.$$

2 Main results

Lemma 1 *Under Assumption A we have the estimate for the following probability,*

$$\mathbb{P}(z_t \leq 0) \leq C \frac{\Delta^{a-\frac{1}{2}}}{e^{\frac{C}{\Delta^{2a-1}}}}.$$

Therefore, the above probability tend to zero faster than any power of Δ .

Proof. Indeed, we have

$$\begin{aligned} \mathbb{P}(z_t \leq 0) &= \mathbb{P}\left(W_t - W_{t_{k+1}} \leq -\frac{(y_{t_k}(1-k\Delta) + \Delta d)^{1-a}}{\sigma(1-a)}\right) \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{C\Delta^{1/2-a}}^{\infty} e^{-\frac{y^2}{2}} dy \\ &\leq C \frac{\Delta^{a-1/2}}{e^{\frac{C}{\Delta^{2a-1}}}}, \end{aligned}$$

where $d = kl - \frac{a\sigma^2}{2}$. To obtain the last inequality one can use problem 9.22, p.112 of [8]. \square

Next we will use the following compact form for our scheme,

$$\begin{aligned} y_t = x_0 &+ \int_0^t (kl - ky_s) ds + \int_0^t \frac{a\sigma^2}{2} (y_s^{2a-1} - y_s^{2a-1}) ds \\ &- \int_t^{t_{k+1}} \left(kl - ky_{t_k} - \frac{a\sigma^2}{2} y_{t_k}^{2a-1} \right) ds + \sigma \int_0^t y_s^a \operatorname{sgn}(z_s) dW_s, \quad t \in (t_k, t_{k+1}]. \end{aligned}$$

Consider the following process,

$$v_t = x_0 + Tkl + \int_0^t \frac{a\sigma^2}{2} y_s^{2a-1} ds + \int_0^t \sigma y_s^a \operatorname{sgn}(z_s) dW_s,$$

Then it is clear that $0 \leq y_t \leq v_t$. We will show that v_t has bounded moments and therefore y_t has also bounded moments.

Lemma 2 (Moment bounds) *Under Assumption A we have the moment bounds,*

$$\mathbb{E}y_t^2 + \mathbb{E}x_t^2 < C,$$

for some $C > 0$

Proof. Consider the stopping time $\theta_R = \inf\{t \geq 0 : v_t > R\}$. Using Ito's formula on $v_{t \wedge \theta_R}^2$ we obtain,

$$v_{t \wedge \theta_R}^2 = (x_0 + Tkl)^2 + \int_0^t \left(\frac{2a\sigma^2}{2} v_{s \wedge \theta_R} y_{s \wedge \theta_R}^{2a-1} + \frac{\sigma^2}{2} y_{s \wedge \theta_R}^{2a} \right) ds + 2\sigma \int_0^t v_{s \wedge \theta_R} y_{s \wedge \theta_R}^a \operatorname{sgn}(z_{s \wedge \theta_R}) dW_s.$$

Taking expectations on both sides and noting that $y_t \leq v_t$, we arrive at

$$\mathbb{E}v_{t \wedge \theta_R}^2 \leq \mathbb{E}(x_0 + Tkl)^2 + C \int_0^t (\mathbb{E}v_{s \wedge \theta_R}^2)^a ds$$

Using now a Gronwall type theorem (see [10], Theorem 1, p. 360), we arrive at

$$\mathbb{E}v_{t \wedge \theta_R}^2 \leq C \quad (3)$$

But $\mathbb{E}v_{t \wedge \theta_R}^2 = \mathbb{E}(v_{t \wedge \theta_R}^2 \mathbb{I}_{\{\theta_R \geq t\}}) + R^p P(\theta_R < t)$. That means that $P(t \wedge \theta_R < t) = P(\theta_R < t) \rightarrow 0$ as $R \rightarrow \infty$ so $t \wedge \theta_R \rightarrow t$ in probability and noting that θ_R increases as R increases we have that $t \wedge \theta_R \rightarrow t$ almost surely too, as $R \rightarrow \infty$. Going back to (3) and using Fatou's lemma we obtain,

$$\mathbb{E}v_t^2 \leq C$$

The same holds for x_t . □

Next we define the process

$$h_t = x_0 + \int_0^t (kl - ky_{\hat{s}})ds + \sigma \int_0^t y_{\hat{s}}^a \text{sgn}(z_s) dW_s$$

We will show that h_t, y_t remain close.

Lemma 3 *We have the following estimates,*

$$\begin{aligned} \mathbb{E}|y_s - y_{\hat{s}}|^2 &\leq C\Delta \text{ for any } s \in [0, T] \\ \mathbb{E}|h_s - y_s|^2 &\leq C\Delta^{2a-1} \text{ for any } s \in [0, T] \\ \mathbb{E}|h_s - y_{\hat{s}}|^2 &\leq C\Delta^{2a-1} \text{ for any } s \in [0, T] \\ \mathbb{E}|h_s|^2 &< A, \text{ for any } s \in [0, T]. \end{aligned}$$

Proof. Using the moment bound for y_t we easily obtain the fact that

$$\mathbb{E}|y_s - y_{\hat{s}}|^2 \leq C\Delta$$

and then

$$\mathbb{E}|h_s - y_s|^2 \leq C\Delta^{2a-1}.$$

Next, we have

$$\mathbb{E}|h_s - y_{t_k}|^2 \leq 2\mathbb{E}|h_s - y_s|^2 + 2\mathbb{E}|y_s - y_{t_k}|^2 \leq C\Delta^{2a-1}.$$

Finally, to get the moment bound for h_t we just use the fact that is close to y_t , i.e.

$$\mathbb{E}h_t^2 \leq 2\mathbb{E}|h_t - y_t|^2 + 2\mathbb{E}y_t^2 \leq C.$$

□

Lemma 4 (Inverse Exponential Moments) *For the true solution x_t it holds the following bound,*

$$\mathbb{E} \left(\exp \left(\frac{C}{x_t^{2(1-a)}} \right) \right) < \infty,$$

for any $C > 0$.

Proof. We first transform our equation with $z_t = x_t^{2(1-a)}$. Then using Ito's formula we deduce that

$$z_t = z_0 + \int_0^t \left(\sigma^2(1-a)(1-2a) - 2(1-a)kz_s + \frac{2(1-a)}{z_s^{\frac{2a-1}{2-2a}}} \right) ds + 2(1-a)\sigma \int_0^t \sqrt{z_s} dW_s,$$

and if we denote by $b(z) = \sigma^2(1-a)(1-2a) - 2(1-a)kz + \frac{2(1-a)}{z^{\frac{2a-1}{2-2a}}}$ then it is easy to see that for any $M > 0$ there exists a c_M such that $b(z) \geq M - c_M z$. We construct now the following CIR process

$$f_t = x_0 + \int_0^t (M - c_M f_s) ds + 2(1-a)\sigma \int_0^t \sqrt{f_s} dW_s.$$

Using a comparison theorem for stochastic differential equations (see [8], prop. 5.2.18) we deduce that $z_t \geq f_t > 0$ (choosing big enough $M > 0$ in order f_t to be strictly positive and also the inverse moment bound holds for f_t) and using the inverse exponential moments of [9] we have the desired result. \square

Theorem 1 *Under Assumption A we have*

$$\mathbb{E}|x_t - y_t|^2 \leq C \Delta^{2a(a-1/2)},$$

and therefore the order of convergence is at least $a(a-1/2)$.

Proof. Using Ito's formula on $|x_\rho - y_\rho|^2$ for some stopping time ρ , we obtain

$$\begin{aligned} \mathbb{E}|x_\rho - h_\rho|^2 &= \int_0^\rho 2k\mathbb{E}(x_s - h_s)(x_s - y_s) + \sigma^2\mathbb{E}(x_s^a - y_s^a \text{sgn}(z_s))^2 ds \\ &\leq \int_0^\rho (2k\mathbb{E}|x_s - h_s|^2 + 2k\mathbb{E}|x_s - h_s||h_s - y_s| + \sigma^2\mathbb{E}(x_s^a - y_s^a \text{sgn}(z_s))^2) ds. \end{aligned}$$

But

$$\begin{aligned} \mathbb{E}(x_s^a - y_s^a \text{sgn}(z_s))^2 &\leq 2\mathbb{E}(x_s^a - y_s^a)^2 + 2\mathbb{E}y_s^{2a}(1 - \text{sgn}(z_s))^2 \\ &= 2\mathbb{E}(x_s^a - y_s^a)^2 + 8\mathbb{P}(z_t < 0)\mathbb{E}(y_s^{2a}|(z_t < 0)) \\ &\leq 2\mathbb{E}(x_s^a - y_s^a)^2 + C \frac{\Delta^{a-1/2}}{e^{\frac{C}{\Delta^{2a-1}}}} \\ &\leq C\Delta^{2a(a-1/2)} + 2\mathbb{E}(x_s^a - h_s^a)^2. \end{aligned}$$

We have used the fact that

$$\mathbb{E}|y_s^a - h_s^a|^2 \leq C\mathbb{E}|y_s - h_s|^{2a} \leq C(\mathbb{E}|y_s - h_s|^2)^a \leq C\Delta^{2a(a-1/2)}.$$

Moreover, by Young inequality,

$$\mathbb{E}|x_s - h_s||h_s - y_s| \leq 1/2\mathbb{E}|x_s - h_s|^2 + 1/2\mathbb{E}|h_s - y_s|^2 \leq C\Delta^{2a-1} + 1/2\mathbb{E}|x_s - h_s|^2.$$

Therefore,

$$\mathbb{E}|x_\rho - h_\rho|^2 \leq C\Delta^{2a(a-1/2)} + \int_0^\rho (3k\mathbb{E}|x_s - h_s|^2 + 2\sigma^2\mathbb{E}(x_s^a - h_s^a)^2) ds$$

Setting now $\gamma_t = \int_0^t \frac{8a^2\sigma^2}{x_s^{2(1-a)} + y_s^{2(1-a)}} ds$ and using the inequality

$$|x^a - y^a| |x^{1-a} + y^{1-a}| \leq 2a|x - y|$$

we have

$$\mathbb{E}|x_\rho - h_\rho|^2 \leq C\Delta^{2a(a-1/2)} + \int_0^\rho \mathbb{E}|x_s - y_s|^2 (3ks + \gamma_s)' ds \quad (4)$$

Define the stopping time

$$\tau_l = \inf\{s \in [0, T] : 3ks + \gamma_s \geq l\}.$$

Now, for $\rho = \tau_l$, we use the change of variables setting $u = 3ks + \gamma_s$ and therefore $s = \tau_u$ obtaining

$$\mathbb{E}(x_{\tau_l} - h_{\tau_l})^2 \leq C\Delta^{2a(a-1/2)} + \int_0^l \mathbb{E}|x_{\tau_u} - h_{\tau_u}|^2 du.$$

Using Gronwall's inequality we obtain,

$$\mathbb{E}|x_{\tau_l} - h_{\tau_l}|^2 \leq Ce^l \Delta^{2a(a-1/2)}. \quad (5)$$

Going back to (4), for $\rho = t \in [0, T]$, we have under the change of variable $u = 3ks + \gamma_s$,

$$\begin{aligned} \mathbb{E}(x_t - h_t)^2 &\leq C\Delta^{2a(a-1/2)} + \mathbb{E} \int_0^{3kT + \gamma_T} |x_{\tau_u} - h_{\tau_u}|^2 du \\ &\leq C\Delta^{2a(a-1/2)} + \int_0^\infty \mathbb{E}(\mathbb{I}_{\{\gamma_T \geq u\}} |x_{\tau_u} - h_{\tau_u}|^2) du. \end{aligned} \quad (6)$$

Using (5) and noting that

$$\mathbb{E}(\mathbb{I}_{\{\gamma_T \geq u\}} |x_{\tau_u} - h_{\tau_u}|^2) = \mathbb{P}(\gamma_T \geq u) \mathbb{E}(|x_{\tau_u} - h_{\tau_u}|^2 | (\gamma_T \geq u))$$

we arrive at

$$\mathbb{E}(x_t - h_t)^2 \leq C\Delta^{2a(a-1/2)} \left(1 + \int_0^\infty \mathbb{P}(\gamma_T \geq u) e^u du \right).$$

Note that,

$$\mathbb{P}(\gamma_T \geq u) \leq \frac{1}{e^u} \mathbb{E}(e^{\gamma_T}),$$

therefore, using the inverse exponential moments of the true solution we deduce that

$$\mathbb{E}(x_t - h_t)^2 \leq C\Delta^{2a(a-1/2)}$$

□

References

- [1] A. Alfonsi, *Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process*, Statistics and Probability Letters, Volume 83, Issue 2, (2013), pp. 602-607.
- [2] L. Andersen, *Simple and efficient simulation of the Heston stochastic volatility model*, Journal of Computational Finance, (2008), Vol. 11, No. 3.
- [3] A. Berkaoui, M. Bossy and A. Diop, *Euler scheme for SDEs with non-Lipchitz diffusion coefficient: strong convergence*, ESAIM 12, (2008), 1-11.
- [4] I. Gyongy and M. Rasonyi, *A note on Euler approximations for SDEs with Holder continuous diffusion coefficients*, Stochastic Processes and their Applications 121 (2011) 2189-2200.
- [5] N. Halidias, *Semi-discrete approximations for stochastic differential equations and applications*, International Journal of Computer Mathematics, (2012), pp. 780-794.
- [6] N. Halidias, *Construction of positivity preserving numerical schemes for a class of multidimensional stochastic differential equations*, Discrete and Continuous Dynamical Systems, 2014.
- [7] N. Halidias, *A novel approach to construct numerical methods for stochastic differential equations*, Numerical Algorithms May 2014, Volume 66, Issue 1, pp 79-87.
- [8] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, 1991.
- [9] T. R. Hurd and A. Kuznetsov, *Explicit formulas for Laplace transforms of stochastic integrals*, Markov Process. Relat. Fields, 14, 277-290 (2008).
- [10] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer, 1991.
- [11] T. Yamada and S. Watanabe *On the uniqueness of solutions of stochastic differential equations.* J. Math. Kyoto Univ. **11**, 155-167, 1971.